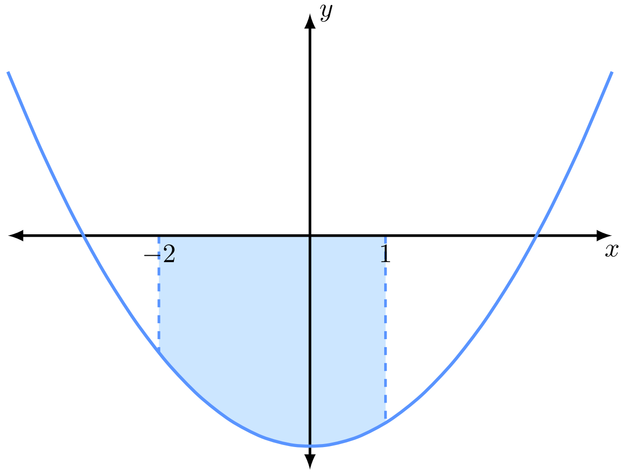
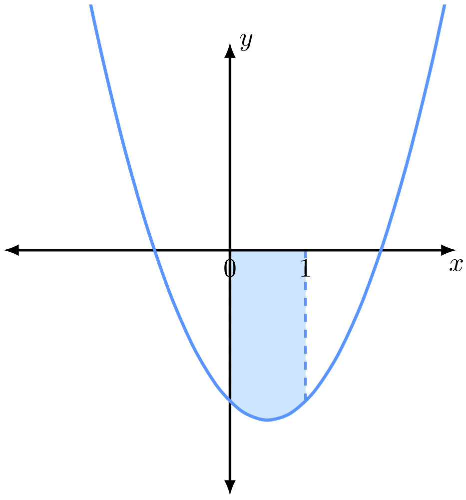
Solution Sheet

1. The area under the curve is given by $-\int_{-2}^{1}(x+3)(x-3)\,\mathrm{d}x$:



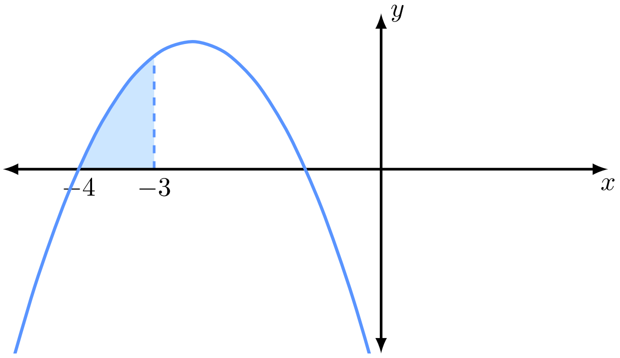
$$A = -\int_{-2}^{1}(x+3)(x-3)\,\mathrm{d}x$$
$$= -\int_{-2}^{1} x^2 - 9\,\mathrm{d}x$$
$$= -\left[\frac{1}{2+1}\times x^{2+1} - 9x\right]_{-2}^{1}$$
$$= -\left[\frac{1}{3}x^3 - 9x\right]_{-2}^{1}$$
$$= -\left(\left(\frac{1}{3}\times 1^3 + -9\times 1\right) - \left(\frac{1}{3}(-2)^3 - 9\times(-2)\right)\right) = 24$$

2. The area under the curve is given by $-\int_0^1 (x+1)(x-2)\,\mathrm{d}x$:



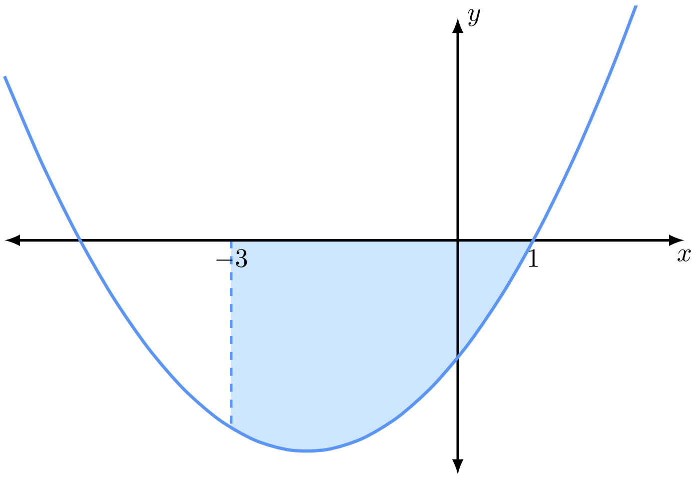
$$
\begin{aligned}
A &= -\int_0^1 (x+1)(x-2)\,\mathrm{d}x \\
&= -\int_0^1 x^2 - x - 2\,\mathrm{d}x \\
&= -\left[\frac{1}{2+1} \times x^{2+1} - \frac{1}{1+1} \times x^{1+1} - 2x\right]_0^1 \\
&= -\left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x\right]_0^1 \\
&= -\left(\left(\frac{1}{3} \times 1^3 + -\frac{1}{2} \times 1^2 + -2 \times 1\right) - \left(\frac{1}{3} \times 0^3 - \frac{1}{2} \times 0^2 - 2 \times 0\right)\right) = \frac{13}{6}
\end{aligned}
$$

3. The area under the curve is given by $\int_{-4}^{-3} -(x+4)(x+1)\,\mathrm{d}x$:



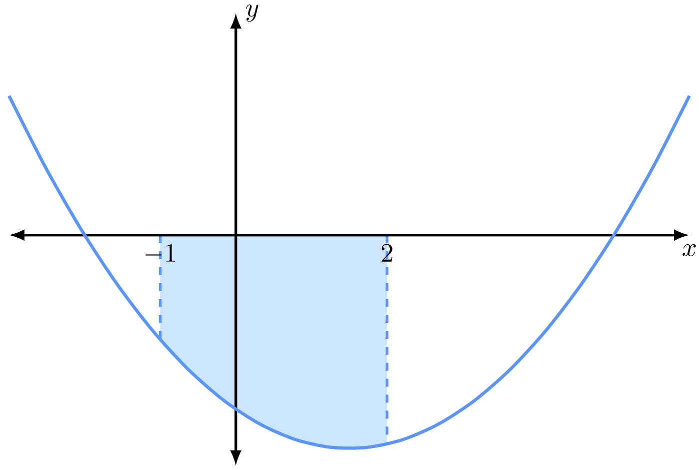
$$\begin{aligned}
A &= \int_{-4}^{-3} -(x+4)(x+1)\,\mathrm{d}x \\
&= \int_{-4}^{-3} -x^2 - 5x - 4\,\mathrm{d}x \\
&= \left[-\frac{1}{3}x^3 - 5 \times \frac{1}{1+1} \times x^{1+1} - 4x\right]_{-4}^{-3} \\
&= \left[-\frac{1}{3}x^3 - \frac{5}{2}x^2 - 4x\right]_{-4}^{-3} \\
&= \left(-\frac{1}{3} \times -3^3 + -\frac{5}{2} \times -3^2 + -4 \times -3\right) - \left(-\frac{1}{3}(-4)^3 - \frac{5}{2}(-4)^2 - 4 \times (-4)\right) = \frac{7}{6}
\end{aligned}$$

4. The area under the curve is given by $-\int_{-3}^{1}(x-1)(x+5)\,\mathrm{d}x$:



$$\begin{aligned}
A &= -\int_{-3}^{1}(x-1)(x+5)\,\mathrm{d}x \\
&= -\int_{-3}^{1} x^2+4x-5\,\mathrm{d}x \\
&= -\left[\frac{1}{2+1}\times x^{2+1}+4\times\frac{1}{1+1}\times x^{1+1}-5x\right]_{-3}^{1} \\
&= -\left[\frac{1}{3}x^3+2x^2-5x\right]_{-3}^{1} \\
&= -\left(\left(\frac{1}{3}\times 1^3+2\times 1^2+-5\times 1\right)-\left(\frac{1}{3}(-3)^3+2(-3)^2-5\times(-3)\right)\right)=\frac{80}{3}
\end{aligned}$$

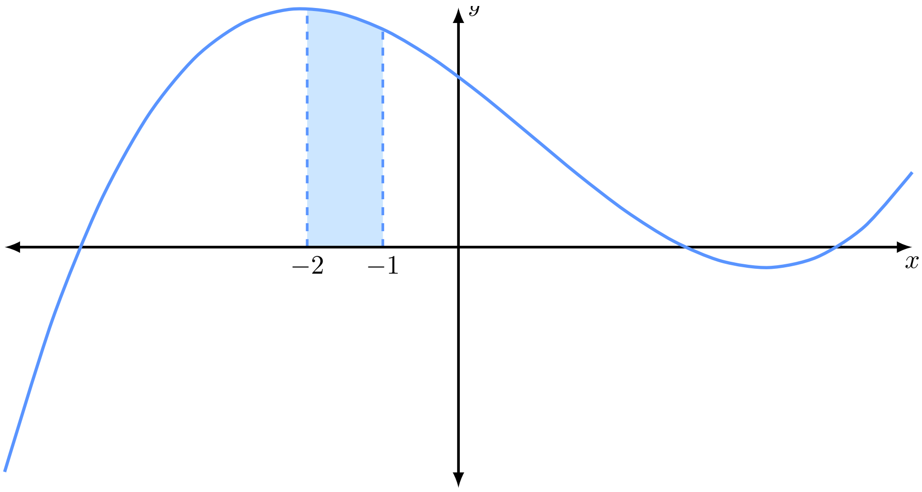
5. The area under the curve is given by $-\int_{-1}^{2}(x+2)(x-5)\,\mathrm{d}x$:



$$\begin{aligned}A &= -\int_{-1}^{2}(x+2)(x-5)\,\mathrm{d}x\\ &= -\int_{-1}^{2}x^2-3x-10\,\mathrm{d}x\\ &= -\left[\frac{1}{2+1}\times x^{2+1}-3\times\frac{1}{1+1}\times x^{1+1}-10x\right]_{-1}^{2}\\ &= -\left[\frac{1}{3}x^3-\frac{3}{2}x^2-10x\right]_{-1}^{2}\\ &= -\left(\left(\frac{1}{3}\times 2^3+-\frac{3}{2}\times 2^2+-10\times 2\right)-\left(\frac{1}{3}(-1)^3-\frac{3}{2}(-1)^2-10\times(-1)\right)\right)=\frac{63}{2}\end{aligned}$$

  student: Any

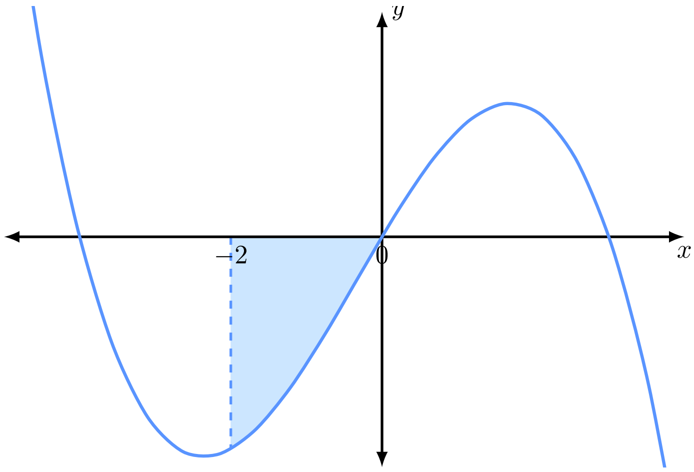
6. Drawing a sketch, we see that the curve is above the x-axis for $-2 \leq x \leq -1$.



Therefore the area is given by

$$\begin{aligned}
&\int_{-2}^{-1} (x+5)(x-3)(x-5)\,\mathrm{d}x \\
&= \int_{-2}^{-1} x^3 - 3x^2 - 25x + 75\,\mathrm{d}x \\
&= \left[\frac{1}{3+1} \times x^{3+1} - 3 \times \frac{1}{2+1} \times x^{2+1} - 25 \times \frac{1}{1+1} \times x^{1+1} + 75x\right]_{-2}^{-1} \\
&= \left[\frac{1}{4}x^4 - x^3 - \frac{25}{2}x^2 + 75x\right]_{-2}^{-1} \\
&= \left(\frac{1}{4} \times -1^4 + -1 \times -1^3 + -\frac{25}{2} \times -1^2 + 75 \times -1\right) - \left(\frac{1}{4}(-2)^4 - (-2)^3 - \frac{25}{2}(-2)^2 - 75 \times 2\right) \\
&= \frac{407}{4}
\end{aligned}$$

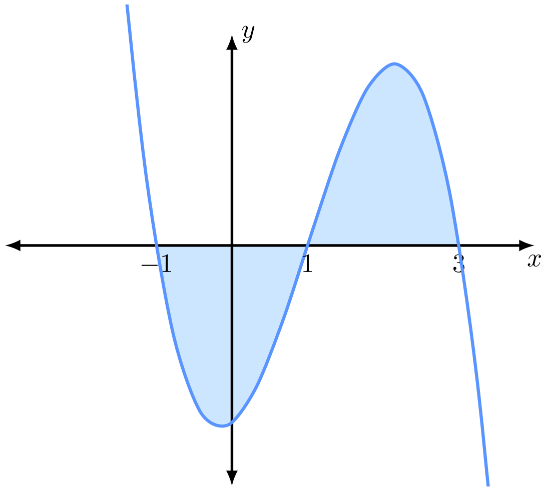
7. Drawing a sketch, we see that the curve is below the x-axis for $-2 \leq x \leq 0$.



Therefore the area is given by

$$
\begin{aligned}
& -\int_{-2}^{0} -(x+4)\,x\,(x-3)\,\mathrm{d}x \\
&= -\int_{-2}^{0} -x^3 - x^2 + 12x\,\mathrm{d}x \\
&= -\left[-\frac{1}{4}x^4 - \frac{1}{3}x^3 + 12 \times \frac{1}{1+1} \times x^{1+1}\right]_{-2}^{0} \\
&= -\left[-\frac{1}{4}x^4 - \frac{1}{3}x^3 + 6x^2\right]_{-2}^{0} \\
&= -\left(\left(-\frac{1}{4} \times 0^4 + -\frac{1}{3} \times 0^3 + 6 \times 0^2\right) - \left(-\frac{1}{4}\left(-2\right)^4 - \frac{1}{3}\left(-2\right)^3 + 6\left(-2\right)^2\right)\right) \\
&= \frac{68}{3}
\end{aligned}
$$

8. Drawing a sketch, we see that the curve is below the x-axis for $-1 \leq x \leq 1$, and above the axis for $1 \leq x \leq 3$.



Therefore the area is given by

$$A = A_1 + A_2$$

where

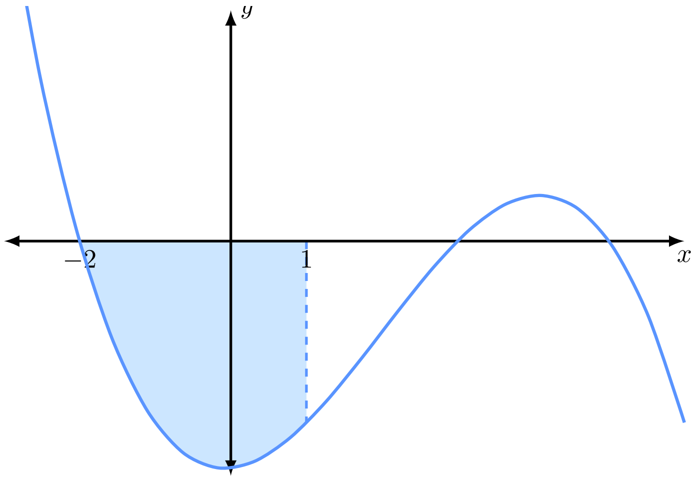
$$A_1 = -\int_{-1}^{1} -(x+1)(x-1)(x-3)\,\mathrm{d}x = -\int_{-1}^{1} -x^3 + 3x^2 + x - 3\,\mathrm{d}x = -\left[-\frac{1}{4}x^4 + x^3 + \frac{1}{2}x^2 - 3x\right]_{-1}^{1} = 4$$

$$A_2 = \int_{1}^{3} -(x+1)(x-1)(x-3)\,\mathrm{d}x = \int_{1}^{3} -x^3 + 3x^2 + x - 3\,\mathrm{d}x = \left[-\frac{1}{4}x^4 + x^3 + \frac{1}{2}x^2 - 3x\right]_{1}^{3} = 4$$

Hence,

$$A = A_1 + A_2 = 4 + 4 = 8$$

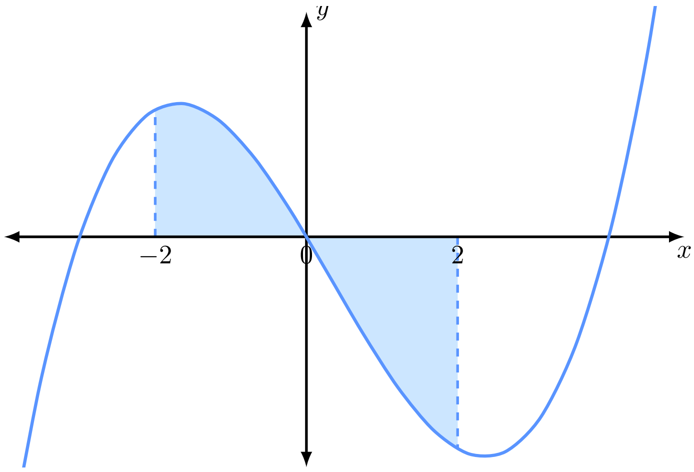
9. Drawing a sketch, we see that the curve is below the x-axis for $-2 \leq x \leq 1$.



Therefore the area is given by

$$
\begin{aligned}
& -\int_{-2}^{1} -(x+2)(x-3)(x-5)\,\mathrm{d}x \\
&= -\int_{-2}^{1} -x^3+6x^2+x-30\,\mathrm{d}x \\
&= -\left[-\frac{1}{4}x^4+6\times\frac{1}{2+1}\times x^{2+1}+\frac{1}{1+1}\times x^{1+1}-30x\right]_{-2}^{1} \\
&= -\left[-\frac{1}{4}x^4+2x^3+\frac{1}{2}x^2-30x\right]_{-2}^{1} \\
&= -\left(\left(-\frac{1}{4}\times 1^4+2\times 1^3+\frac{1}{2}\times 1^2+-30\times 1\right)-\left(-\frac{1}{4}(-2)^4+2(-2)^3+\frac{1}{2}(-2)^2-30\times(-2)\right)\right) \\
&= \frac{279}{4}
\end{aligned}
$$

10. Drawing a sketch, we see that the curve is above the x-axis for $-2 \leq x \leq 0$, and below the axis for $0 \leq x \leq 2$.



Therefore the area is given by

$$A = A_1 + A_2$$

where

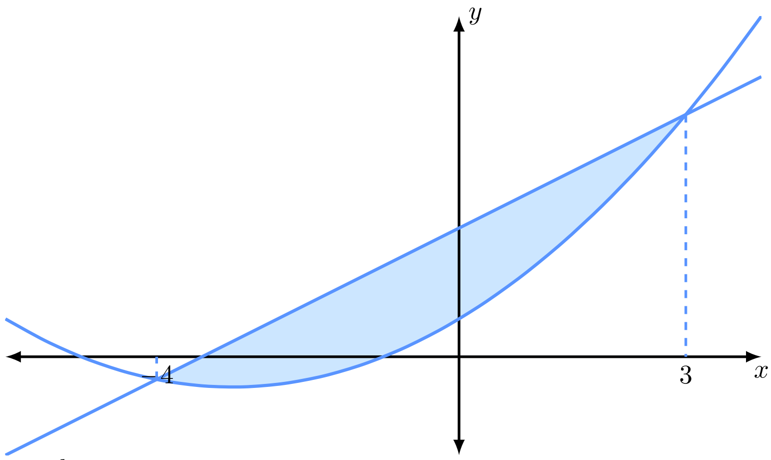
$$A_1 = \int_{-2}^{0} (x+3)\,x\,(x-4)\,\mathrm{d}x = \int_{-2}^{0} x^3 - x^2 - 12x\,\mathrm{d}x = \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - 6x^2\right]_{-2}^{0} = \frac{52}{3}$$

$$A_2 = -\int_{0}^{2} (x+3)\,x\,(x-4)\,\mathrm{d}x = -\int_{0}^{2} x^3 - x^2 - 12x\,\mathrm{d}x = -\left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - 6x^2\right]_{0}^{2} = \frac{68}{3}$$

Hence,

$$A = A_1 + A_2 = \frac{52}{3} + \frac{68}{3} = 40$$

11. First, we label the sketch:



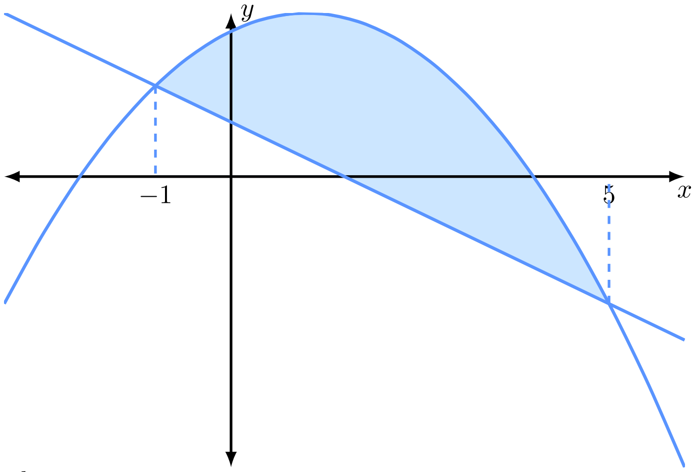
The curve and line intersect when:

$$\begin{aligned}(x+5)\,(x+1) &= 5x + 17 \\ 0 &= -x + 12 - x^2 \\ 0 &= (4+x)\,(3-x) \\ x &= -4,\ 3\end{aligned}$$

We see that the top curve is $y_1(x) = 5x + 17$, and the bottom curve is $y_2(x) = (x+5)(x+1)$. Therefore the area is

$$\begin{aligned}\int_{-4}^{3} (y_1 - y_2)\,\mathrm{d}x &= \int_{-4}^{3} (5x+17) - (x+5)(x+1)\,\mathrm{d}x \\ &= \int_{-4}^{3} -x + 12 - x^2\,\mathrm{d}x \\ &= \left[-\frac{1}{1+1} \times x^{1+1} + 12x - \frac{1}{3}x^3\right]_{-4}^{3} \\ &= \left[-\frac{1}{2}x^2 + 12x - \frac{1}{3}x^3\right]_{-4}^{3} \\ &= \left(-\frac{1}{2} \times 3^2 + 12 \times 3 + -\frac{1}{3} \times 3^3\right) - \left(-\frac{1}{2}(-4)^2 - 12 \times 4 - \frac{1}{3}(-4)^3\right) \\ &= \frac{343}{6}\end{aligned}$$

12. First, we label the sketch:



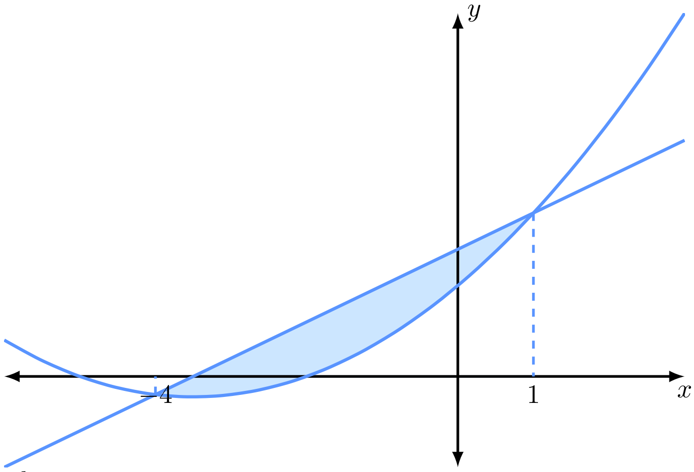
The curve and line intersect when:

$$\begin{aligned}-(x+2)(x-4) &= -2x + 3 \\ 0 &= -4x - 5 + x^2 \\ 0 &= (x+1)(x-5) \\ x &= -1,\ 5\end{aligned}$$

We see that the top curve is $y_1(x) = -(x+2)(x-4)$, and the bottom curve is $y_2(x) = -2x+3$. Therefore the area is

$$\begin{aligned}
\int_{-1}^{5} (y_1 - y_2)\,\mathrm{d}x &= \int_{-1}^{5} -(x+2)(x-4) - (-2x+3)\,\mathrm{d}x \\
&= \int_{-1}^{5} -x^2 + 4x + 5\,\mathrm{d}x \\
&= \left[-\frac{1}{3}x^3 + 4 \times \frac{1}{1+1} \times x^{1+1} + 5x\right]_{-1}^{5} \\
&= \left[-\frac{1}{3}x^3 + 2x^2 + 5x\right]_{-1}^{5} \\
&= \left(-\frac{1}{3} \times 5^3 + 2 \times 5^2 + 5 \times 5\right) - \left(-\frac{1}{3}(-1)^3 + 2(-1)^2 - 5\right) \\
&= 36
\end{aligned}$$

13. First, we label the sketch:



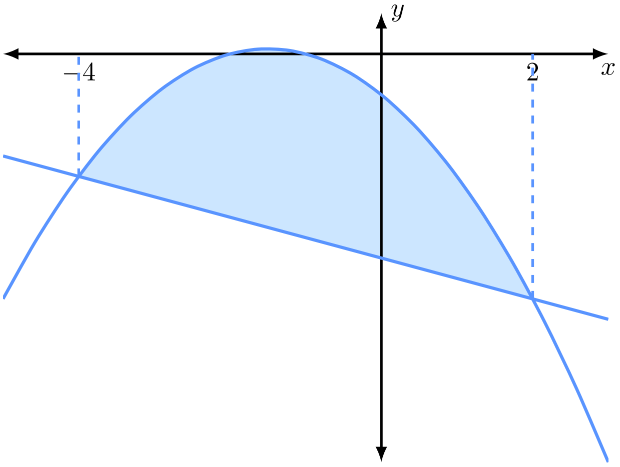
The curve and line intersect when:

$$\begin{aligned}
(x+5)(x+2) &= 4x + 14 \\
0 &= -3x + 4 - x^2 \\
0 &= (4+x)(1-x) \\
x &= -4,\ 1
\end{aligned}$$

We see that the top curve is $y_1(x) = 4x + 14$, and the bottom curve is $y_2(x) = (x+5)(x+2)$. Therefore the area is

$$\begin{aligned}
\int_{-4}^{1} (y_1 - y_2)\,\mathrm{d}x &= \int_{-4}^{1} (4x+14) - (x+5)(x+2)\,\mathrm{d}x \\
&= \int_{-4}^{1} -3x + 4 - x^2\,\mathrm{d}x \\
&= \left[-3 \times \frac{1}{1+1} \times x^{1+1} + 4x - \frac{1}{3}x^3\right]_{-4}^{1} \\
&= \left[-\frac{3}{2}x^2 + 4x - \frac{1}{3}x^3\right]_{-4}^{1} \\
&= \left(-\frac{3}{2} \times 1^2 + 4 + -\frac{1}{3} \times 1^3\right) - \left(-\frac{3}{2}(-4)^2 - 4 \times 4 - \frac{1}{3}(-4)^3\right) \\
&= \frac{125}{6}
\end{aligned}$$

14. First, we label the sketch:



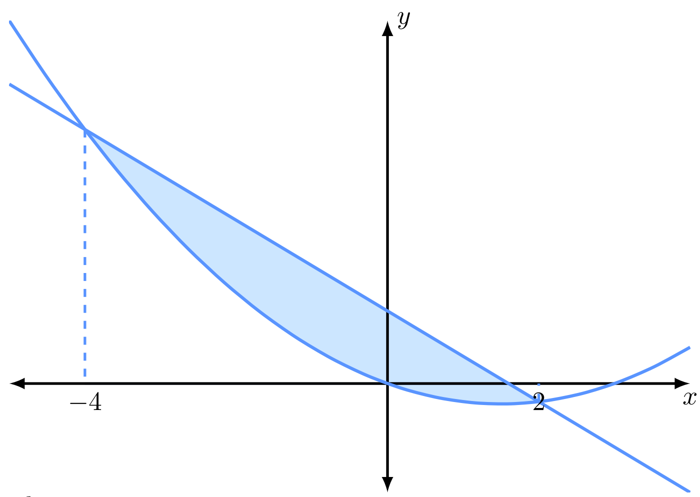
The curve and line intersect when:

$$\begin{aligned}
-(x+2)(x+1) &= -x - 10 \\
0 &= 2x - 8 + x^2 \\
0 &= (x+4)(x-2) \\
x &= -4,\ 2
\end{aligned}$$

We see that the top curve is $y_1(x) = -(x+2)(x+1)$, and the bottom curve is $y_2(x) = -x - 10$. Therefore the area is

$$\begin{aligned}\int_{-4}^{2} (y_1 - y_2)\,\mathrm{d}x &= \int_{-4}^{2} -(x+2)(x+1) - (-x-10)\,\mathrm{d}x \\ &= \int_{-4}^{2} -x^2 - 2x + 8\,\mathrm{d}x \\ &= \left[-\frac{1}{3}x^3 - 2 \times \frac{1}{1+1} \times x^{1+1} + 8x\right]_{-4}^{2} \\ &= \left[-\frac{1}{3}x^3 - x^2 + 8x\right]_{-4}^{2} \\ &= \left(-\frac{1}{3} \times 2^3 + -1 \times 2^2 + 8 \times 2\right) - \left(-\frac{1}{3}(-4)^3 - (-4)^2 - 8 \times 4\right) \\ &= 36\end{aligned}$$

15. First, we label the sketch:



The curve and line intersect when:

$$\begin{aligned}x(x-3) &= -5x + 8 \\ 0 &= -2x + 8 - x^2 \\ 0 &= (4+x)(2-x) \\ x &= -4,\ 2\end{aligned}$$

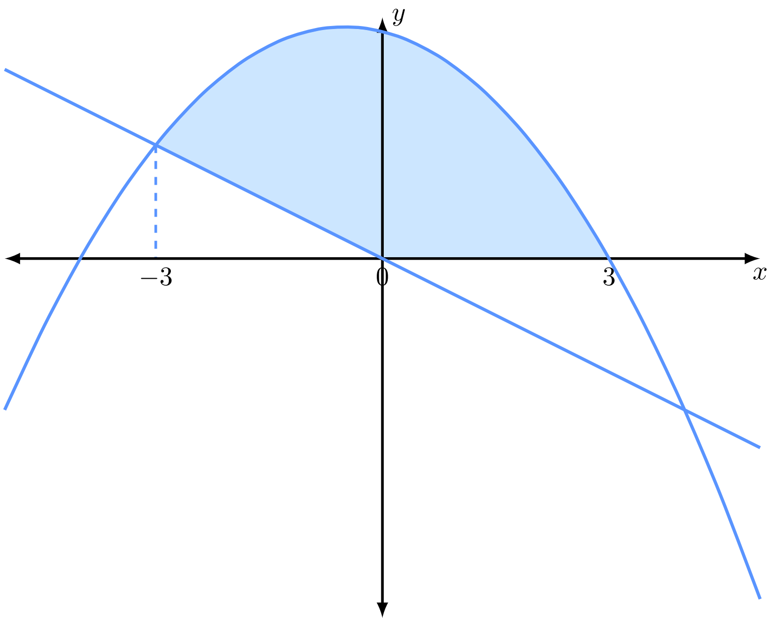
We see that the top curve is $y_1(x) = -5x + 8$, and the bottom curve is $y_2(x) = x(x-3)$. Therefore the area is

$$\begin{aligned}
\int_{-4}^{2} (y_1 - y_2)\,\mathrm{d}x &= \int_{-4}^{2} (-5x+8) - x\,(x-3)\,\mathrm{d}x \\
&= \int_{-4}^{2} -2x + 8 - x^2\,\mathrm{d}x \\
&= \left[-2 \times \frac{1}{1+1} \times x^{1+1} + 8x - \frac{1}{3}x^3\right]_{-4}^{2} \\
&= \left[-x^2 + 8x - \frac{1}{3}x^3\right]_{-4}^{2} \\
&= \left(-1 \times 2^2 + 8 \times 2 + -\frac{1}{3} \times 2^3\right) - \left(-\,(-4)^2 - 8 \times 4 - \frac{1}{3}\,(-4)^3\right) \\
&= 36
\end{aligned}$$

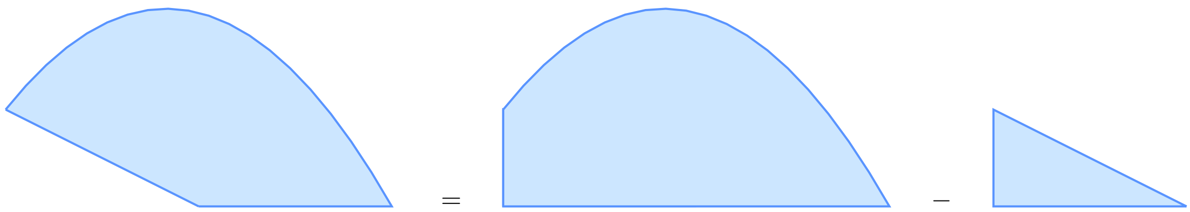
16. First, we label the sketch. The curve intersects the x-axis at $x = -4,\ 3$, and the line intersects the x-axis at $x = 0$. The curve and line intersect when:

$$\begin{aligned}
-\,(x+4)\,(x-3) &= -2x \\
0 &= -x - 12 + x^2 \\
0 &= (x+3)\,(x-4) \\
x &= -3,\ 4
\end{aligned}$$

and substitution into either equation gives the intersection points as $(-3, 6), (4, -8)$. By looking at the sketch, we see that the region is bounded by the points $(0,0), (3,0)$ and $(-3,6)$.



By dropping a line from the intersection point to the x-axis, we make a triangle. The triangle has base 3 and height 6, so has area $A_1 = 9$.
If we add the triangular area to the shaded region, we get a region bounded by the curve and the x-axis, which we can find by integration.



The curved area is given by

$$\begin{aligned} A_2 &= \int_{-3}^{3} -(x+4)(x-3)\,\mathrm{d}x \\ &= \int_{-3}^{3} -x^2 - x + 12\,\mathrm{d}x \\ &= \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 12x\right]_{-3}^{3} = 54 \end{aligned}$$

Therefore the shaded area is

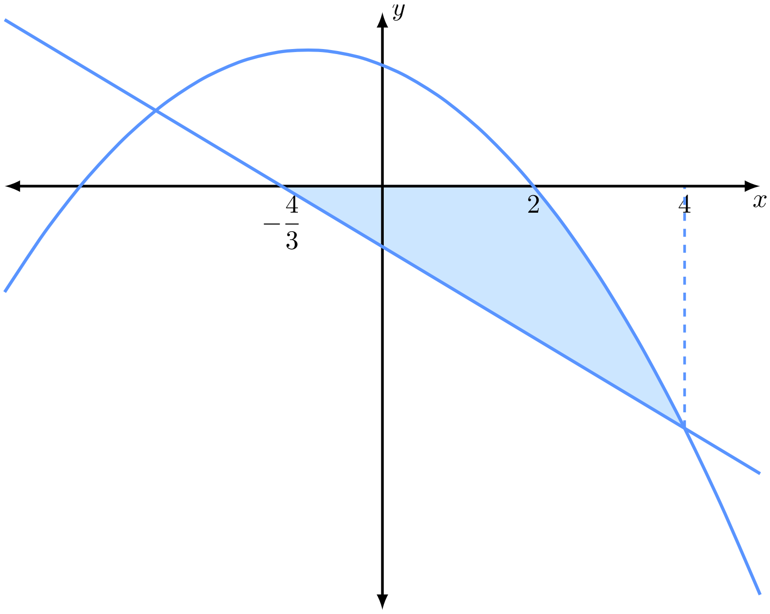
$$A = A_2 - A_1 = 54 - 9 = 45$$

.

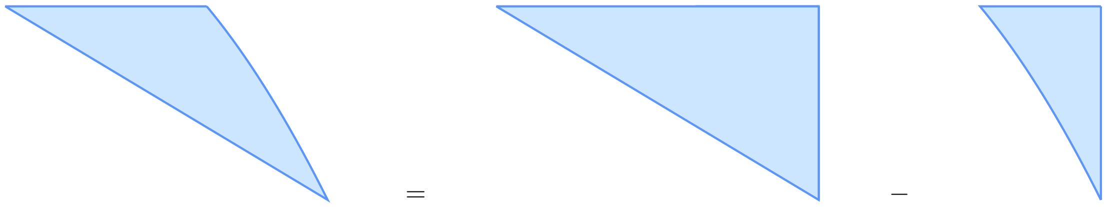
17. First, we label the sketch. The curve intersects the x-axis at $x = -4,\ 2$, and the line intersects the x-axis at $x = -\frac{4}{3}$. The curve and line intersect when:

$$\begin{aligned} -(x+4)(x-2) &= -3x - 4 \\ 0 &= -x - 12 + x^2 \\ 0 &= (x+3)(x-4) \\ x &= -3,\ 4 \end{aligned}$$

and substitution into either equation gives the intersection points as $(-3, 5), (4, -16)$. By looking at the sketch, we see that the region is bounded by the points $(-\frac{4}{3}, 0), (2, 0)$ and $(4, -16)$.



By dropping a line from the intersection point to the x-axis, we make a triangle which contains the shaded region. The triangle has base $\frac{16}{3}$ and height 16, so has area $A_1 = \frac{128}{3}$.



The remaining curved area is bounded by the curve and the x-axis, which we can find by integration. The curved region is below the axis, so the area is given by

$$\begin{aligned} A_2 &= -\int_2^4 -(x+4)(x-2)\,\mathrm{d}x \\ &= -\int_2^4 -x^2 - 2x + 8\,\mathrm{d}x \\ &= -\left[-\frac{1}{3}x^3 - x^2 + 8x\right]_2^4 = \frac{44}{3} \end{aligned}$$

Therefore the shaded area is

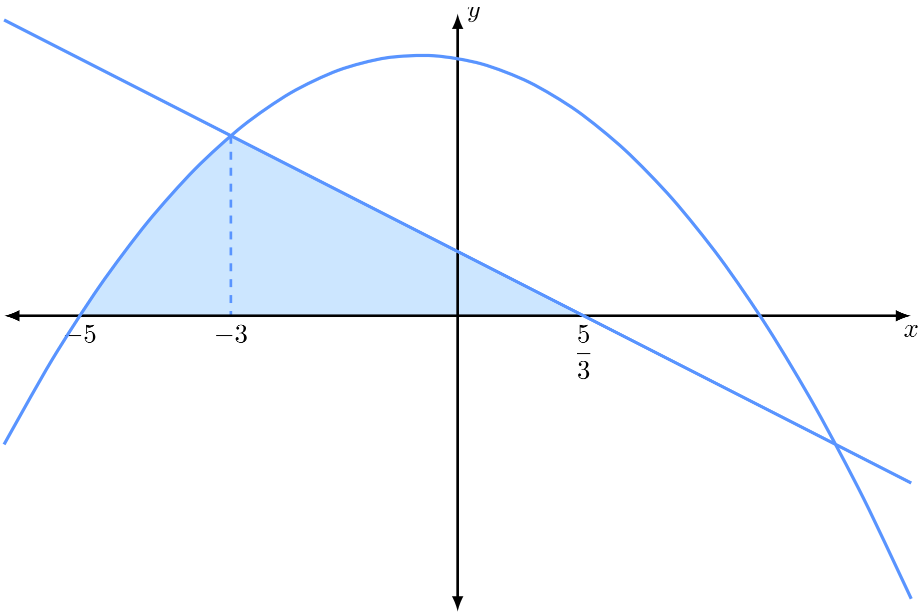
$$A = A_1 - A_2 = \frac{128}{3} - \frac{44}{3} = 28$$

.

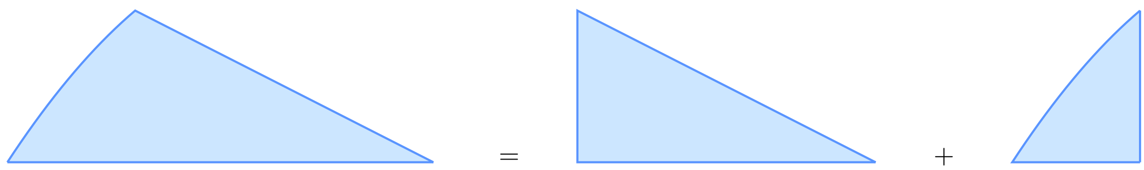
18. First, we label the sketch. The curve intersects the x-axis at $x = -5,\ 4$, and the line intersects the x-axis at $x = \frac{5}{3}$. The curve and line intersect when:

$$\begin{aligned} -(x+5)(x-4) &= -3x+5 \\ 0 &= -2x - 15 + x^2 \\ 0 &= (x+3)(x-5) \\ x &= -3,\ 5 \end{aligned}$$

and substitution into either equation gives the intersection points as $(-3, 14), (5, -10)$. By looking at the sketch, we see that the region is bounded by the points $(\frac{5}{3}, 0), (-5, 0)$ and $(-3, 14)$.



If we drop a line from the intersection point to the x-axis, we divide the region into two areas: a triangle, and a region which is bounded by the curve and the x-axis. The triangle has base $\frac{14}{3}$ and height 14, so has area $A_1 = \frac{98}{3}$.



The curved area is given by

$$\begin{aligned} A_2 &= \int_{-5}^{-3} -(x+5)(x-4)\,\mathrm{d}x \\ &= \int_{-5}^{-3} -x^2 - x + 20\,\mathrm{d}x \\ &= \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 20x\right]_{-5}^{-3} = \frac{46}{3} \end{aligned}$$

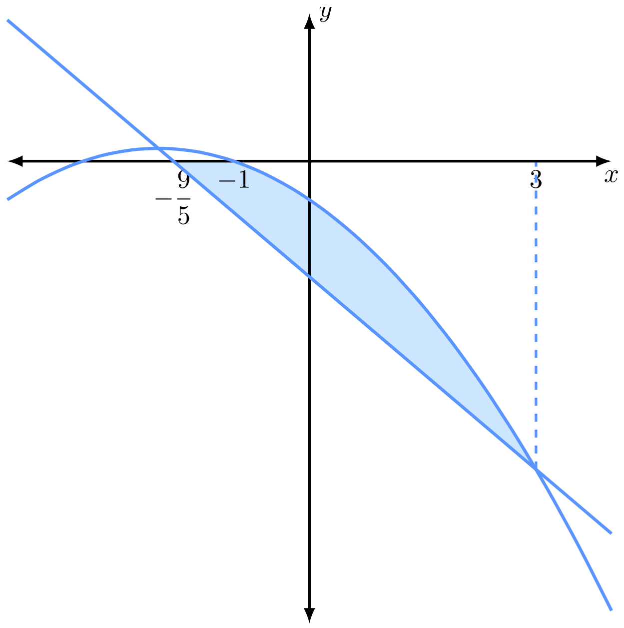
Therefore the shaded area is

$$A = A_1 + A_2 = \frac{98}{3} + \frac{46}{3} = 48.$$

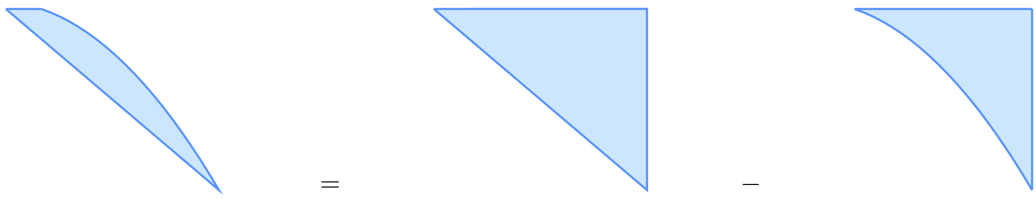
19. First, we label the sketch. The curve intersects the x-axis at $x = -3,\ -1$, and the line intersects the x-axis at $x = -\frac{9}{5}$. The curve and line intersect when:

$$\begin{aligned} -(x+3)(x+1) &= -5x-9 \\ 0 &= -x-6+x^2 \\ 0 &= (x+2)(x-3) \\ x &= -2,\ 3 \end{aligned}$$

and substitution into either equation gives the intersection points as $(-2,1),(3,-24)$. By looking at the sketch, we see that the region is bounded by the points $(-\frac{9}{5},0),(-1,0)$ and $(3,-24)$.



By dropping a line from the intersection point to the x-axis, we make a triangle which contains the shaded region. The triangle has base $\frac{24}{5}$ and height 24, so has area $A_1 = \frac{288}{5}$.



The remaining curved area is bounded by the curve and the x-axis, which we can find by integration. The curved region is below the axis, so the area is given by

$$\begin{aligned} A_2 &= -\int_{-1}^{3} -(x+3)(x+1)\,\mathrm{d}x \\ &= -\int_{-1}^{3} -x^2-4x-3\,\mathrm{d}x \\ &= -\left[-\frac{1}{3}x^3-2x^2-3x\right]_{-1}^{3} = \frac{112}{3} \end{aligned}$$

Therefore the shaded area is

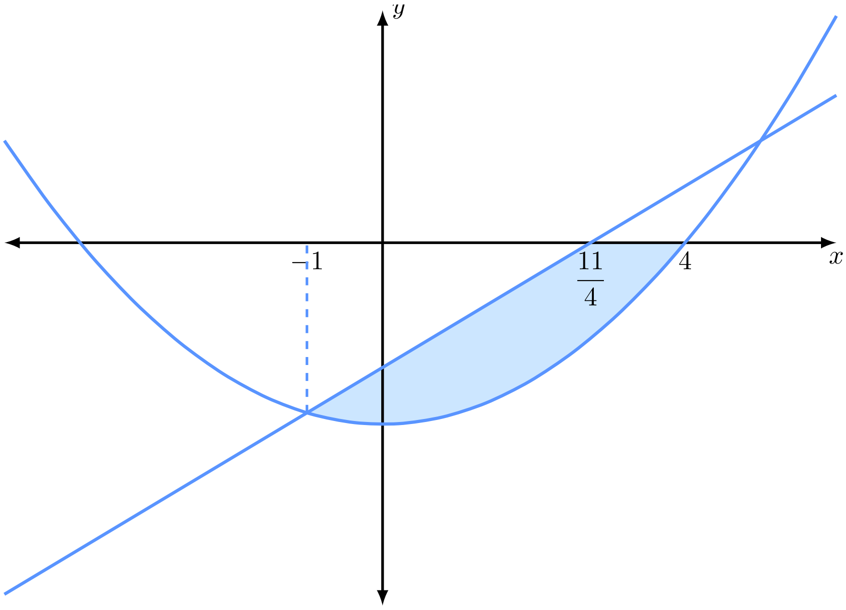
$$A = A_1 - A_2 = \frac{288}{5} - \frac{112}{3} = \frac{304}{15}$$

.

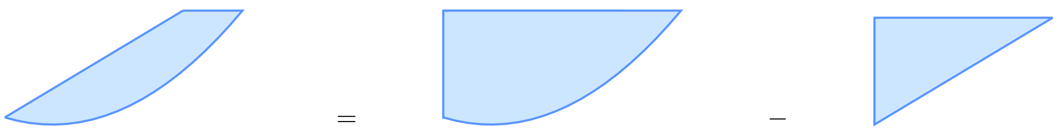
20. First, we label the sketch. The curve intersects the x-axis at $x = -4,\ 4$, and the line intersects the x-axis at $x = \frac{11}{4}$. The curve and line intersect when:

$$\begin{aligned}(x+4)(x-4) &= -11 + 4x \\ 0 &= 5 + 4x - x^2 \\ 0 &= (1+x)(5-x) \\ x &= -1,\ 5\end{aligned}$$

and substitution into either equation gives the intersection points as $(-1,-15),(5,9)$. By looking at the sketch, we see that the region is bounded by the points $(\frac{11}{4},0),(4,0)$ and $(-1,-15)$.



By dropping a line from the intersection point to the x-axis, we make a triangle. The triangle has base $\frac{15}{4}$ and height 15, so has area $A_1 = \frac{225}{8}$.
If we add the triangular area to the shaded region, we get a region bounded by the curve and the x-axis, which we can find by integration.



The curved region is below the axis, so the area is given by

$$\begin{aligned}A_2 &= -\int_{-1}^{4} (x+4)(x-4)\,\mathrm{d}x \\ &= -\int_{-1}^{4} x^2 - 16\,\mathrm{d}x \\ &= -\left[\frac{1}{3}x^3 - 16x\right]_{-1}^{4} = \frac{175}{3}\end{aligned}$$

Therefore the shaded area is

$$A = A_2 - A_1 = \frac{175}{3} - \frac{225}{8} = \frac{725}{24}$$

.

21. The curves intesect when

$$\begin{aligned}(x+4)(x-1) &= -(x+4)(x+1)(x-1)\\ -(x+1) &= 1\\ x &= -2\end{aligned}$$

$$\begin{aligned}A_1 &= \int_{-4}^{-2} (\text{top curve} - \text{bottom curve})\,\mathrm{d}x\\ &= \int_{-4}^{-2} (x+4)(x-1) + (x+4)(x+1)(x-1)\,\mathrm{d}x\\ &= \int_{-4}^{-2} 5x^2 + 2x - 8 + x^3\,\mathrm{d}x\\ &= \left[\frac{5}{3}x^3 + x^2 - 8x + \frac{1}{4}x^4\right]_{-4}^{-2}\\ &= \frac{16}{3}\end{aligned}$$

$$\begin{aligned}A_2 &= \int_{-2}^{1} (\text{top curve} - \text{bottom curve})\,\mathrm{d}x\\ &= \int_{-2}^{1} -(x+4)(x+1)(x-1) - (x+4)(x-1)\,\mathrm{d}x\\ &= \int_{-2}^{1} -x^3 - 5x^2 - 2x + 8\,\mathrm{d}x\\ &= \left[-\frac{1}{4}x^4 - \frac{5}{3}x^3 - x^2 + 8x\right]_{-2}^{1}\\ &= \frac{63}{4}\end{aligned}$$

Therefore,

$$A = A_1 + A_2 = \frac{16}{3} + \frac{63}{4} = \frac{253}{12}$$

22. The curves intesect when

$$\begin{aligned}-(x+4)(x-1) &= (x+4)(x+1)(x-1)\\ -(x+1) &= 1\\ x &= -2\end{aligned}$$

$$\begin{aligned}A_1 &= \int_{-4}^{-2} (\text{top curve} - \text{bottom curve})\,\mathrm{d}x\\ &= \int_{-4}^{-2} (x+4)(x+1)(x-1) + (x+4)(x-1)\,\mathrm{d}x\\ &= \int_{-4}^{-2} x^3 + 5x^2 + 2x - 8\,\mathrm{d}x\\ &= \left[\frac{1}{4}x^4 + \frac{5}{3}x^3 + x^2 - 8x\right]_{-4}^{-2}\\ &= \frac{16}{3}\end{aligned}$$

$$\begin{aligned}A_2 &= \int_{-2}^{1} (\text{top curve} - \text{bottom curve})\,\mathrm{d}x\\ &= \int_{-2}^{1} -(x+4)(x-1) - (x+4)(x+1)(x-1)\,\mathrm{d}x\\ &= \int_{-2}^{1} -5x^2 - 2x + 8 - x^3\,\mathrm{d}x\\ &= \left[-\frac{5}{3}x^3 - x^2 + 8x - \frac{1}{4}x^4\right]_{-2}^{1}\\ &= \frac{63}{4}\end{aligned}$$

Therefore,

$$A = A_1 + A_2 = \frac{16}{3} + \frac{63}{4} = \frac{253}{12}$$

23. The curves intesect when

$$
\begin{aligned}
-(x+1)(x-4) &= -(x+1)(x-1)(x-4) \\
x-1 &= 1 \\
x &= 2
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= \int_{-1}^{2} (\text{top curve} - \text{bottom curve})\,\mathrm{d}x \\
&= \int_{-1}^{2} -(x+1)(x-4) + (x+1)(x-1)(x-4)\,\mathrm{d}x \\
&= \int_{-1}^{2} -5x^2 + 2x + 8 + x^3\,\mathrm{d}x \\
&= \left[-\frac{5}{3}x^3 + x^2 + 8x + \frac{1}{4}x^4\right]_{-1}^{2} \\
&= \frac{63}{4}
\end{aligned}
$$

$$
\begin{aligned}
A_2 &= \int_{2}^{4} (\text{top curve} - \text{bottom curve})\,\mathrm{d}x \\
&= \int_{2}^{4} -(x+1)(x-1)(x-4) + (x+1)(x-4)\,\mathrm{d}x \\
&= \int_{2}^{4} -x^3 + 5x^2 - 2x - 8\,\mathrm{d}x \\
&= \left[-\frac{1}{4}x^4 + \frac{5}{3}x^3 - x^2 - 8x\right]_{2}^{4} \\
&= \frac{16}{3}
\end{aligned}
$$

Therefore,

$$A = A_1 + A_2 = \frac{63}{4} + \frac{16}{3} = \frac{253}{12}$$

24. The curves intesect when

$$
\begin{aligned}
(x+1)(x-5) &= -(x+1)(x-3)(x-5) \\
-(x-3) &= 1 \\
x &= 2
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= \int_{-1}^{2} (\text{top curve} - \text{bottom curve})\,\mathrm{d}x \\
&= \int_{-1}^{2} (x+1)(x-5) + (x+1)(x-3)(x-5)\,\mathrm{d}x \\
&= \int_{-1}^{2} -6x^2 + 3x + 10 + x^3\,\mathrm{d}x \\
&= \left[-2x^3 + \frac{3}{2}x^2 + 10x + \frac{1}{4}x^4\right]_{-1}^{2} \\
&= \frac{81}{4}
\end{aligned}
$$

$$
\begin{aligned}
A_2 &= \int_{2}^{5} (\text{top curve} - \text{bottom curve})\,\mathrm{d}x \\
&= \int_{2}^{5} -(x+1)(x-3)(x-5) - (x+1)(x-5)\,\mathrm{d}x \\
&= \int_{2}^{5} -x^3 + 6x^2 - 3x - 10\,\mathrm{d}x \\
&= \left[-\frac{1}{4}x^4 + 2x^3 - \frac{3}{2}x^2 - 10x\right]_{2}^{5} \\
&= \frac{81}{4}
\end{aligned}
$$

Therefore,

$$A = A_1 + A_2 = \frac{81}{4} + \frac{81}{4} = \frac{81}{2}$$

25. The curves intesect when

$$\begin{aligned}(x+4)(x-5) &= -(x+4)(x-3)(x-5)\\ -(x-3) &= 1\\ x &= 2\end{aligned}$$

$$\begin{aligned}A_1 &= \int_{-4}^{2} (\text{top curve} - \text{bottom curve})\,\mathrm{d}x\\ &= \int_{-4}^{2} (x+4)(x-5) + (x+4)(x-3)(x-5)\,\mathrm{d}x\\ &= \int_{-4}^{2} -3x^2 - 18x + 40 + x^3\,\mathrm{d}x\\ &= \left[-x^3 - 9x^2 + 40x + \frac{1}{4}x^4\right]_{-4}^{2}\\ &= 216\end{aligned}$$

$$\begin{aligned}A_2 &= \int_{2}^{5} (\text{top curve} - \text{bottom curve})\,\mathrm{d}x\\ &= \int_{2}^{5} -(x+4)(x-3)(x-5) - (x+4)(x-5)\,\mathrm{d}x\\ &= \int_{2}^{5} -x^3 + 3x^2 + 18x - 40\,\mathrm{d}x\\ &= \left[-\frac{1}{4}x^4 + x^3 + 9x^2 - 40x\right]_{2}^{5}\\ &= \frac{135}{4}\end{aligned}$$

Therefore,

$$A = A_1 + A_2 = 216 + \frac{135}{4} = \frac{999}{4}$$

26. First we find the equations of the normals. The gradient of a tangent to $y = \frac{1}{2}x^2$ is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x$$

Therefore the gradient of the normal at $x = -1$ is 1, and at $x = 1$ is -1

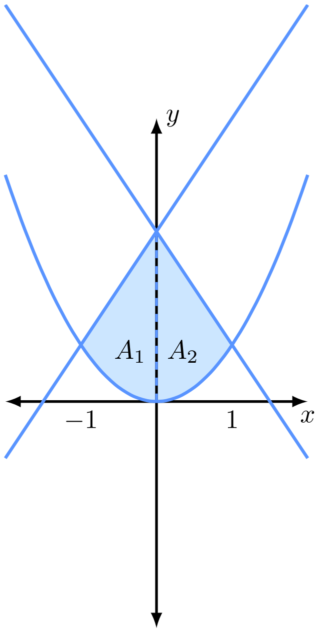
$$\begin{aligned}y - y_1 &= m(x - x_1)\\ y - \frac{1}{2} &= x - 1 \times (-1)\\ y &= x + \frac{3}{2}\end{aligned}$$

$$\begin{aligned}y - y_1 &= m(x - x_1)\\ y - \frac{1}{2} &= -(x-)\\ y &= -x + \frac{3}{2}\end{aligned}$$

The lines intersect when

$$\begin{aligned}x + \frac{3}{2} &= -x + \frac{3}{2}\\ x &= 0\end{aligned}$$

Now we sketch the curve:



$$
\begin{aligned}
A_1 &= \int_{-1}^{0} (\text{top curve} - \text{bottom curve})\, \mathrm{d}x \\
&= \int_{-1}^{0} \left(x + \frac{3}{2}\right) - \frac{1}{2}x^2 \, \mathrm{d}x \\
&= \int_{-1}^{0} x + \frac{3}{2} - \frac{1}{2}x^2 \, \mathrm{d}x \\
&= \left[\frac{1}{2}x^2 + \frac{3}{2}x - \frac{1}{6}x^3\right]_{-1}^{0} \\
&= \frac{5}{6}
\end{aligned}
$$

By symmetry,

$$A_1 = A_2 \implies A = A_1 + A_2 = \frac{5}{6} + \frac{5}{6} = \frac{5}{3}$$

27. First we find the equations of the normals. The gradient of a tangent to $y = \frac{1}{2}x^2$ is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x$$

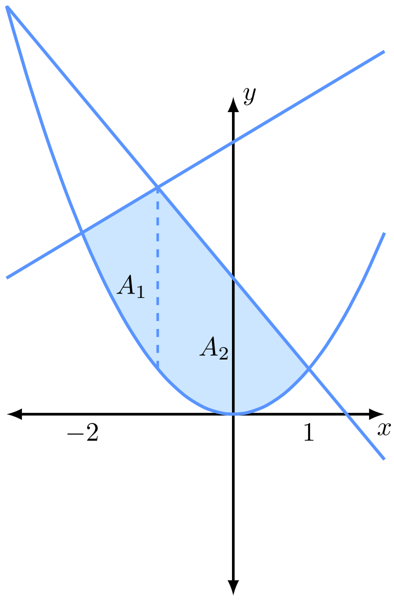
Therefore the gradient of the normal at $x = -2$ is $\frac{1}{2}$, and at $x = 1$ is -1

$$
\begin{aligned}
y - y_1 &= m\,(x - x_1) \\
y - 2 &= \frac{1}{2}\,(x - (-2)) \\
y &= \frac{1}{2}x + 3
\end{aligned}
\qquad\qquad
\begin{aligned}
y - y_1 &= m\,(x - x_1) \\
y - \frac{1}{2} &= -\,(x-) \\
y &= -x + \frac{3}{2}
\end{aligned}
$$

The lines intersect when

$$\frac{1}{2}x + 3 = -x + \frac{3}{2}$$
$$x = -1$$

Now we sketch the curve:



$$\begin{aligned} A_1 &= \int_{-2}^{-1} (\text{top curve} - \text{bottom curve})\,\mathrm{d}x \\ &= \int_{-2}^{-1} \left(\frac{1}{2}x + 3\right) - \frac{1}{2}x^2\,\mathrm{d}x \\ &= \int_{-2}^{-1} \frac{1}{2}x + 3 - \frac{1}{2}x^2\,\mathrm{d}x \\ &= \left[\frac{1}{4}x^2 + 3x - \frac{1}{6}x^3\right]_{-2}^{-1} \\ &= \frac{13}{12} \end{aligned}$$

$$\begin{aligned} A_2 &= \int_{-1}^{1} (\text{top curve} - \text{bottom curve})\,\mathrm{d}x \\ &= \int_{-1}^{1} \left(-x + \frac{3}{2}\right) - \frac{1}{2}x^2\,\mathrm{d}x \\ &= \int_{-1}^{1} -x + \frac{3}{2} - \frac{1}{2}x^2\,\mathrm{d}x \\ &= \left[-\frac{1}{2}x^2 + \frac{3}{2}x - \frac{1}{6}x^3\right]_{-1}^{1} \\ &= \frac{8}{3} \end{aligned}$$

Therefore,

$$A = A_1 + A_2 = \frac{13}{12} + \frac{8}{3} = \frac{15}{4}$$

28. First we find the equations of the normals. The gradient of a tangent to $y = \frac{1}{4}x^2$ is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}x$$

Therefore the gradient of the normal at $x = -2$ is 1, and at $x = 1$ is -2

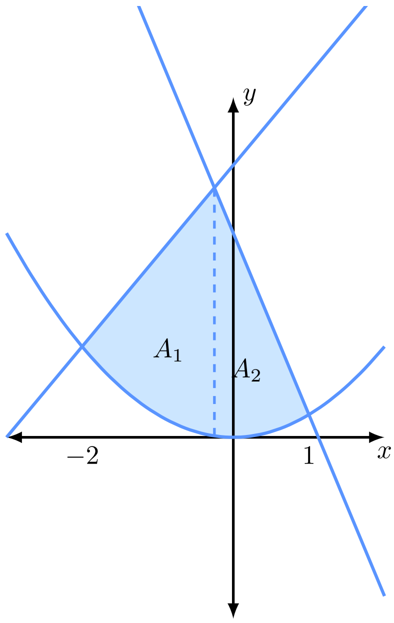
$$y - y_1 = m(x - x_1)$$
$$y - 1 = x - 1 \times (-2)$$
$$y = x + 3$$

$$y - y_1 = m(x - x_1)$$
$$y - \frac{1}{4} = -2(x - 1)$$
$$y = -2x + \frac{9}{4}$$

The lines intersect when

$$x + 3 = -2x + \frac{9}{4}$$
$$x = -\frac{1}{4}$$

Now we sketch the curve:



$$\begin{aligned} A_1 &= \int_{-2}^{-\frac{1}{4}} (\text{top curve} - \text{bottom curve})\, \mathrm{d}x \\ &= \int_{-2}^{-\frac{1}{4}} (x+3) - \frac{1}{4}x^2\, \mathrm{d}x \\ &= \int_{-2}^{-\frac{1}{4}} x + 3 - \frac{1}{4}x^2\, \mathrm{d}x \\ &= \left[\frac{1}{2}x^2 + 3x - \frac{1}{12}x^3\right]_{-2}^{-\frac{1}{4}} \\ &= \frac{2009}{768} \end{aligned}$$

$$\begin{aligned} A_2 &= \int_{-\frac{1}{4}}^{1} (\text{top curve} - \text{bottom curve})\, \mathrm{d}x \\ &= \int_{-\frac{1}{4}}^{1} \left(-2x + \frac{9}{4}\right) - \frac{1}{4}x^2\, \mathrm{d}x \\ &= \int_{-\frac{1}{4}}^{1} -2x + \frac{9}{4} - \frac{1}{4}x^2\, \mathrm{d}x \\ &= \left[-x^2 + \frac{9}{4}x - \frac{1}{12}x^3\right]_{-\frac{1}{4}}^{1} \\ &= \frac{1375}{768} \end{aligned}$$

Therefore,

$$A = A_1 + A_2 = \frac{2009}{768} + \frac{1375}{768} = \frac{141}{32}$$

29. First we find the equations of the normals. The gradient of a tangent to $y = \frac{1}{4}x^2$ is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}x$$

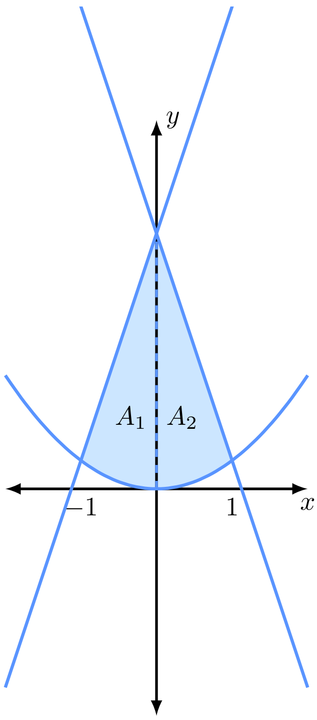
Therefore the gradient of the normal at $x = -1$ is 2, and at $x = 1$ is -2

$$\begin{aligned} y - y_1 &= m\left(x - x_1\right) \\ y - \frac{1}{4} &= 2\left(x - (-1)\right) \\ y &= 2x + \frac{9}{4} \end{aligned} \qquad \begin{aligned} y - y_1 &= m\left(x - x_1\right) \\ y - \frac{1}{4} &= -2\left(x - 1\right) \\ y &= -2x + \frac{9}{4} \end{aligned}$$

The lines intersect when

$$\begin{aligned} 2x + \frac{9}{4} &= -2x + \frac{9}{4} \\ x &= 0 \end{aligned}$$

Now we sketch the curve:



$$\begin{aligned} A_1 &= \int_{-1}^{0} (\text{top curve} - \text{bottom curve})\,\mathrm{d}x \\ &= \int_{-1}^{0} \left(2x + \frac{9}{4}\right) - \frac{1}{4}x^2\,\mathrm{d}x \\ &= \int_{-1}^{0} 2x + \frac{9}{4} - \frac{1}{4}x^2\,\mathrm{d}x \\ &= \left[x^2 + \frac{9}{4}x - \frac{1}{12}x^3\right]_{-1}^{0} \\ &= \frac{7}{6} \end{aligned}$$

By symmetry,

$$A_1 = A_2 \implies A = A_1 + A_2 = \frac{7}{6} + \frac{7}{6} = \frac{7}{3}$$

30. First we find the equations of the normals. The gradient of a tangent to $y = \frac{1}{3}x^2$ is

$$\frac{dy}{dx} = \frac{2}{3}x$$

Therefore the gradient of the normal at $x = -3$ is $\frac{1}{2}$, and at $x = 2$ is $-\frac{3}{4}$

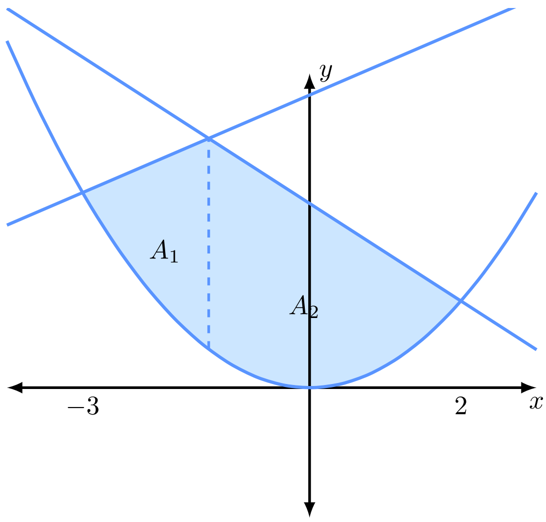
$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= \frac{1}{2}(x - (-3)) \\ y &= \frac{1}{2}x + \frac{9}{2} \end{aligned}$$

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - \frac{4}{3} &= -\frac{3}{4}(x - 2) \\ y &= -\frac{3}{4}x + \frac{17}{6} \end{aligned}$$

The lines intersect when

$$\begin{aligned} \frac{1}{2}x + \frac{9}{2} &= -\frac{3}{4}x + \frac{17}{6} \\ x &= -\frac{4}{3} \end{aligned}$$

Now we sketch the curve:



$$\begin{aligned} A_1 &= \int_{-3}^{-\frac{4}{3}} (\text{top curve} - \text{bottom curve})\, dx \\ &= \int_{-3}^{-\frac{4}{3}} \left(\frac{1}{2}x + \frac{9}{2}\right) - \frac{1}{3}x^2\, dx \\ &= \int_{-3}^{-\frac{4}{3}} \frac{1}{2}x + \frac{9}{2} - \frac{1}{3}x^2\, dx \\ &= \left[\frac{1}{4}x^2 + \frac{9}{2}x - \frac{1}{9}x^3\right]_{-3}^{-\frac{4}{3}} \\ &= \frac{2875}{972} \end{aligned}$$

$$\begin{aligned} A_2 &= \int_{-\frac{4}{3}}^{2} (\text{top curve} - \text{bottom curve})\, dx \\ &= \int_{-\frac{4}{3}}^{2} \left(-\frac{3}{4}x + \frac{17}{6}\right) - \frac{1}{3}x^2\, dx \\ &= \int_{-\frac{4}{3}}^{2} -\frac{3}{4}x + \frac{17}{6} - \frac{1}{3}x^2\, dx \\ &= \left[-\frac{3}{8}x^2 + \frac{17}{6}x - \frac{1}{9}x^3\right]_{-\frac{4}{3}}^{2} \\ &= \frac{3625}{486} \end{aligned}$$

Therefore,

$$A = A_1 + A_2 = \frac{2875}{972} + \frac{3625}{486} = \frac{125}{12}$$