Those questions in this exercise where the context is drawn from real life involve working with mathematical models.

① The first part of a race track is a bend. As the leading car travels round the bend its position, in metres, is modelled by:

$$\mathbf{r} = 2t^2\mathbf{i} + 8t\mathbf{j}$$

where t is in seconds.

(i) Find an expression for the velocity of the car.

(ii) Find the position of the car when $t = 0, 1, 2, 3$ and 4.

Use this information to sketch the path of the car.

(iii) Find the velocity of the car when $t = 0, 1, 2, 3$ and 4.

Add vectors to your sketch to represent these velocities.

(iv) Find the speed of the car as it leaves the bend at $t = 5$.

② As a boy slides down a slide his position vector in metres at time t is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 16 - 4t \\ 20 - 5t \end{pmatrix}.$$

Find his velocity and acceleration.

③ Calculate the magnitude and direction of the acceleration of a particle that moves so that its position vector in metres is given by

$$\mathbf{r} = (8t - 2t^2)\mathbf{i} + (6 + 4t - t^2)\mathbf{j}$$

where t is the time in seconds.

④ A rocket moves with a velocity (in $\mathrm{m\,s^{-1}}$) modelled by

$$\mathbf{v} = \tfrac{1}{10}t\mathbf{i} + \tfrac{1}{10}t^2\mathbf{j}$$

where $\mathbf{i}$ and $\mathbf{j}$ are horizontal and vertical unit vectors respectively and t is in seconds. Find

(i) an expression for its position vector relative to its starting position at time t

(ii) the displacement of the rocket after 10 s of its flight.

⑤ A particle is initially at rest at the origin. It experiences an acceleration given by

$$\mathbf{a} = 4t\mathbf{i} + (6 - 2t)\mathbf{j}$$

Find expressions for the velocity and position of the particle at time t.

⑥ A particle has position vector $\mathbf{r} = 3t^2\mathbf{i} - 4t\mathbf{j}$ at time t. Find the angle between its position vector and its direction of motion at time $t = 2$.

⑦ While a hockey ball is being hit it experiences an acceleration (in $\mathrm{m\,s^{-2}}$) modelled by

$$\mathbf{a} = 1000[6t(t - 0.2)\mathbf{i} + t(t - 0.2)\mathbf{j}]$$

$$\text{for } 0 \leqslant t \leqslant 0.2 \text{ (in seconds)}$$

and $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along and perpendicular to the side of the pitch.

The ball is initially at rest. At $t = 0.2$ it loses contact with the hockey stick.

Find its speed when $t = 0.2$.

⑧ A speedboat is initially moving at $5\,\text{m s}^{-1}$ on a bearing of 135°.

(i) Express the initial velocity as a vector in terms of **i** and **j**, which are vectors east and north respectively.

The boat then begins to accelerate with an acceleration modelled by

$$\mathbf{a} = 0.1t\mathbf{i} + 0.3t\mathbf{j} \text{ in m s}^{-2}.$$

(ii) Find the velocity of the boat 10 s after it begins to accelerate and its displacement over the 10 s period.

⑨ A girl throws a ball and, t seconds after she releases it, its position in metres relative to the point where she is standing is modelled by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15t \\ 2+16t-5t^2 \end{pmatrix}$$

where the directions are horizontal and vertical.

(i) Find expressions for the velocity and acceleration of the ball at time t.

(ii) The vertical component of the velocity is zero when the ball is at its highest point. Find the time taken for the ball to reach this point.

(iii) When the ball hits the ground the vertical component of its position vector is zero. What is the speed of the ball when it hits the ground?

(iv) Find the equation of the trajectory of the ball.

⑩ The position (in metres) of a tennis ball t seconds after leaving a racquet is modelled by

$$\mathbf{r} = 20t\mathbf{i} + (2 + t - 5t^2)\mathbf{j}$$

where **i** and **j** are horizontal and vertical unit vectors.

(i) Find the position of the tennis ball when $t = 0, 0.2, 0.4, 0.6$ and 0.8.

Use these to sketch the path of the ball.

(ii) Find an expression for the velocity of the tennis ball.

Use this to find the velocity of the ball when $t = 0.2$.

(iii) Find the acceleration of the ball.

(iv) Find the equation of the trajectory of the ball.

⑪ An owl is initially perched on a tree. It then goes for a short flight which ends when it dives onto a mouse on the ground. The position vector (in metres) of the owl t seconds into its flight is modelled by

$$\mathbf{r} = t^2(6 - t)\mathbf{i} + (12.5 + 4.5t^2 - t^3)\mathbf{j}$$

where the foot of the tree is taken to be the origin and the unit vectors **i** and **j** are horizontal and vertical.

(i) Draw a graph showing the bird's flight.

(ii) For how long (in s) is the owl in flight?

(iii) Find the speed of the owl when it catches the mouse and the angle that its flight makes with the horizontal at that instant.

(iv) Show that the owl's acceleration is never zero during the flight.

⑫ A particle P of mass 5 units moves under the action of a force $\begin{pmatrix} 10 \\ -5 \\ 20 \end{pmatrix}$.

Initially P has velocity $\begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}$ and is at the point with position vector $\begin{pmatrix} 2 \\ -5 \\ 8 \end{pmatrix}$.

Find, at time $t = 4$,

(i) the speed of P

(ii) the position vector of P.

⑬ Ship A is 5 km due west of ship B and is travelling on a course 035° at a constant but unknown speed $v\,\text{km h}^{-1}$. Ship B is travelling at a constant $10\,\text{km h}^{-1}$ on a course 300°.

(i) Write the velocity of each ship in terms of unit vectors **i** and **j** with directions east and north.

(ii) Find the position vector of each ship at time t hours, relative to the starting position of ship A.

The ships are on a collision course.

(iii) Find the speed of ship A.

(iv) How much time elapses before the collision occurs?

⑭ A particle of mass 0.5 kg is acted on by a force, in newtons,

$$\mathbf{F} = t^2\mathbf{i} + 2t\mathbf{j}.$$

The particle is initially at rest at the origin and t is measured in seconds.

(i) Find the acceleration of the particle at time t.

(ii) Find the velocity of the particle at time t.

(iii) Find the position vector of the particle at time t.

(iv) Find the speed and direction of motion of the particle at time $t = 2$.

⑮ The position vector $\mathbf{r}$ of a moving particle at time t after the start of the motion is given by

$$\mathbf{r} = (5 + 20t)\mathbf{i} + \left(95 + 10t - 5t^2\right)\mathbf{j}.$$

(i) Find the initial velocity of the particle.

At time $t = T$ the particle is moving at right angles to its initial direction of motion.

(ii) Find the value of T and the distance of the particle from its initial position at this time.

⑯ A small, delicate microchip which is initially at rest is to be moved by a robot arm so that it is placed *gently* onto a horizontal assembly bench. Two mathematical models have been proposed for the motion which will be programmed into the robot. In each model the unit of length is the centimetre and time is measured in seconds. The unit vectors $\mathbf{i}$ and $\mathbf{j}$ have directions which are horizontal and vertical respectively and the origin is the point O on the surface of the bench, as shown in Figure 18.16.

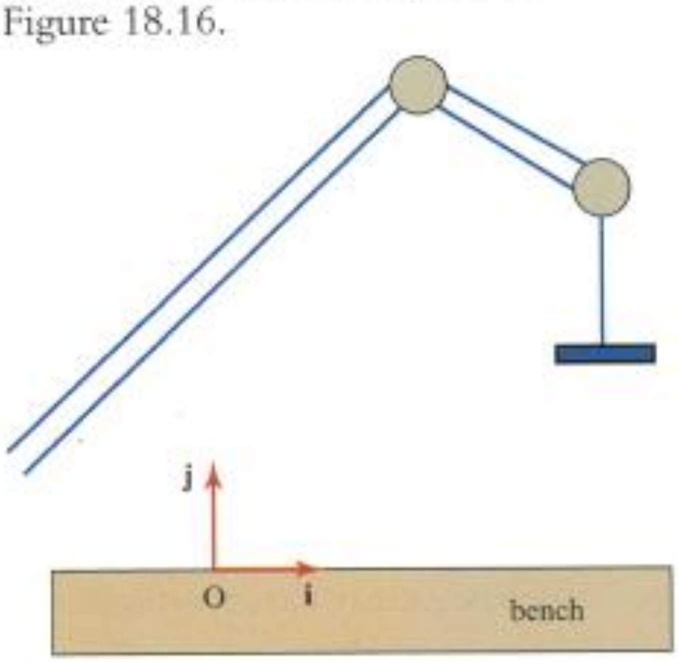


Figure 18.16

Model A for the position vector of the microchip at time t is

$$\mathbf{r}_A = 5t^2\mathbf{i} + \left(16 - 4t^2\right)\mathbf{j} \quad (t \geqslant 0).$$

(i) How far above the bench is the microchip initially (i.e. when $t = 0$)?

(ii) Show that this model predicts that the microchip reaches the bench after 2 s and state the horizontal distance moved in this time.

(iii) Calculate the predicted horizontal and vertical components of velocity when $t = 0$ and $t = 2$.

Model B for the position vector at time t of the microchip is

$$\mathbf{r}_B = \left(15t^2 - 5t^3\right)\mathbf{i} + \left(16 - 24t^2 + 16t^3 - 3t^4\right)\mathbf{j}$$
$$(t \geqslant 0).$$

(iv) Show that model B predicts the same positions for the microchip at $t = 0$ and $t = 2$ as model A.

(v) Calculate the predicted horizontal and vertical components of velocity for the microchip at $t = 0$ and $t = 2$ from model B and comment, with brief reasons, on which model you think describes the more suitable motion.

⑱ A hawk H and a sparrow S fly with constant velocities $\mathbf{v}_H = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v}_S = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ respectively.

At time $t = 0$ the position vector of the hawk is $\mathbf{r}_H = \mathbf{j}$ and that of the sparrow is $\mathbf{r}_S = 5\mathbf{i}$.

(i) Write down the velocity of H relative to S.

(ii) Write down the position vector of H relative to S at time t.

(iii) Find the time at which the birds are closest together and determine their least distance apart.

⑲ A particle moves in the xy plane and at time t has acceleration $\mathbf{a} = 2\mathbf{i}$. Initially the particle is at (3, 1) and is moving with velocity $\mathbf{u} = -2\mathbf{i} + \mathbf{j}$. Show that the path of the particle is a parabola and find its Cartesian equation.

⑳ At time t two points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$ respectively, where

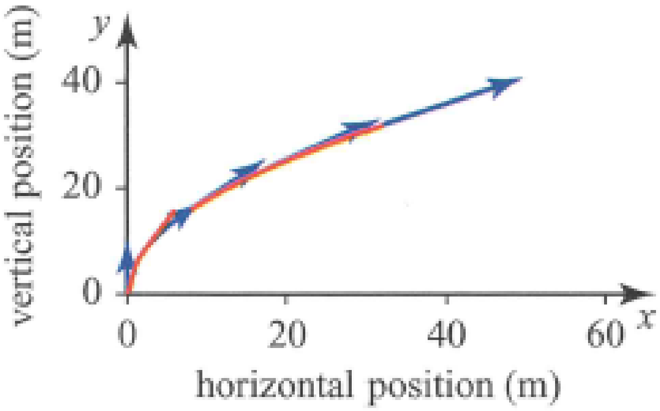
$$\mathbf{p} = 2\mathbf{i} + \cos\omega t\mathbf{j} + \sin\omega t\mathbf{k}$$

$$\mathbf{q} = \sin\omega t\mathbf{i} - \cos\omega t\mathbf{j} + 3\mathbf{k}$$

where ω is a constant. Find $\mathbf{r}$, the position vector of P relative to Q, and $\mathbf{v}$ the velocity of P relative to Q.

1 (i) $\mathbf{v} = 4t\mathbf{i} + 8\mathbf{j}$

(ii)



(0, 0); (2, 8); (8, 16); (18, 24); (32, 32)

(iii) $8\mathbf{j}$; $4\mathbf{i} + 8\mathbf{j}$; $8\mathbf{i} + 8\mathbf{j}$; $12\mathbf{i} + 8\mathbf{j}$; $16\mathbf{i} + 8\mathbf{j}$

(iv) $21.5\,\text{m s}^{-1}$

2 $\mathbf{v} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}$; $\mathbf{a} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

3 $4.47\,\text{m s}^{-2}$; $-153°$

4 (i) $\frac{1}{20}t^2\mathbf{i} + \frac{1}{30}t^3\mathbf{j}$

(ii) $5\mathbf{i} + 33\frac{1}{3}\mathbf{j}$

5 $\mathbf{v} = 2t^2\mathbf{i} + \left(6t - t^2\right)\mathbf{j}$;

$\mathbf{r} = \frac{2}{3}t^3\mathbf{i} + \left(3t^2 - \frac{1}{3}t^3\right)\mathbf{j}$

6 $15.3°$

7 $8.11\,\text{m s}^{-1}$

8 (i) initial velocity = $3.54\mathbf{i} - 3.54\mathbf{j}$

(ii) $\mathbf{v} = 8.54\mathbf{i} + 11.46\mathbf{j}$; $\mathbf{r} = 52.0\mathbf{i} + 14.6\mathbf{j}$

9 (i) $\mathbf{v} = \begin{pmatrix} 15 \\ 16 - 10t \end{pmatrix}$;

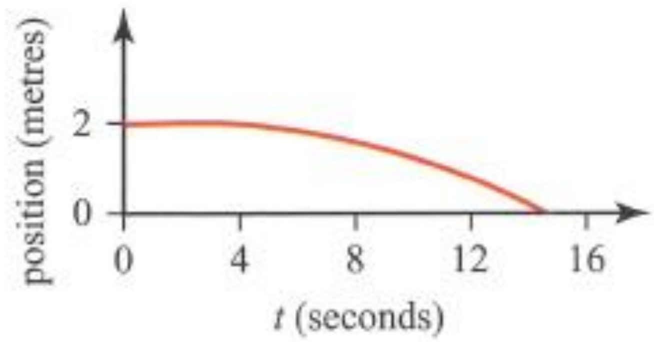
$\mathbf{a} = \begin{pmatrix} 0 \\ -10 \end{pmatrix}$

(ii) $1.6\,\text{s}$

(iii) $22.8\,\text{m s}^{-1}$

(iv) $y = 2 + \frac{16}{15}x - \frac{1}{45}x^2$

10 (i)



(0, 2); (4, 2); (8, 1.6); (12, 0.8); (16, –0.4)

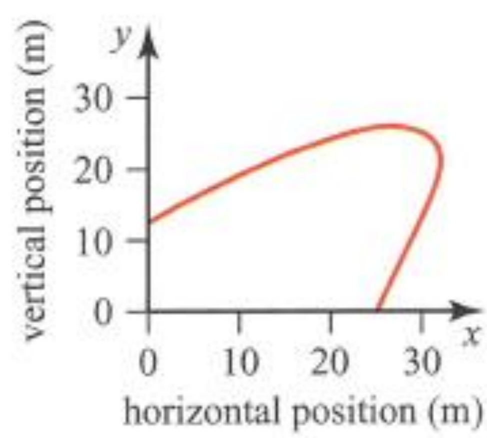
(ii) $\mathbf{v} = 20\mathbf{i} + (1 - 10t)\,\mathbf{j}$;

$20\mathbf{i} - \mathbf{j}$

(iii) $-10\mathbf{j}$

(iv) $y = 2 + \frac{1}{20}x - \frac{1}{80}x^2$

11 (i)



(ii) $5\,\text{s}$

(iii) $33.5\,\text{m s}^{-1}$, $63.4°$ (or $116.6°$)

(iv)

$\mathbf{a} = (12 - 6t)\,\mathbf{i} + (9 - 6t)\,\mathbf{j}$ is never 0

12 (i) $16.9\,\mathrm{m\,s^{-1}}$

(ii)

$$\mathbf{r} = \begin{pmatrix} 2 - 3t + t^2 \\ -5 + 2t - 0.5t^2 \\ 8 + 2t^2 \end{pmatrix}, \begin{pmatrix} 6 \\ -5 \\ 40 \end{pmatrix}$$

13 (i) A: $v \sin 35°\mathbf{i} + v\cos 35°\mathbf{j}$
B: $-8.66\mathbf{i} + 5\mathbf{j}$

(ii) A: $vt \sin 35°\mathbf{i} + vt\cos 35°\mathbf{j}$;
B: $(5 - 8.66t)\mathbf{i} + 5t\mathbf{j}$

(iii) $6.1\,\mathrm{km\,h^{-1}}$

(iv) 0.4111… hours = 24.7 min

14 (i) $2t^2\mathbf{i} + 4t\mathbf{j}$

(ii) $\frac{2}{3}t^3\mathbf{i} + 2t^2\mathbf{j}$

(iii) $\frac{1}{6}t^4\mathbf{i} + \frac{2}{3}t^3\mathbf{j}$

(iv) speed $9.61\,\mathrm{m\,s^{-1}}$; 56.3° to $\mathbf{i}$ direction.

15 (i) $20\mathbf{i}+10\mathbf{j}$

(ii) $T = 5$ s; 125 m

16 (i) 16 cm

(ii) 20 cm

(iii) $0\,\mathrm{cm\,s^{-1}}, 0\,\mathrm{cm\,s^{-1}}$;
$20\,\mathrm{cm\,s^{-1}}, -16\,\mathrm{cm\,s^{-1}}$

(iv) $t = 0, \mathbf{r}_A = \mathbf{r}_B = 16\mathbf{j}$
$t = 2, \mathbf{r}_A = \mathbf{r}_B = 20\mathbf{i}$

(v) All components are zero, so model B is better.

18 (i) $2\mathbf{i} + \mathbf{j} + \mathbf{k}$

(ii) $(2t - 5)\mathbf{i} + (t + 1)\mathbf{j} + t\mathbf{k}$

(iii) 1.5 s; 3.5 m

19 $x = 6 - 4y + y^2$

20 $\mathbf{r} = (2 - \sin \omega t)\mathbf{i} + 2\cos \omega t\,\mathbf{j} + (\sin \omega t - 3)\mathbf{k}$;
$\mathbf{v} = -\omega\cos\omega t\,\mathbf{i} - 2\omega\sin\omega t\,\mathbf{j} + \omega\cos\omega t\,\mathbf{k}$