Dodgy Proofs

Stage: 5 Challenge Level: ★★★

Consider these dodgy proofs. Although the results are obviously wrong, where, precisely, do the 'proofs' break down? There are five fun introductory questions, four main questions and four extension questions. Good luck!

INTRODUCTORY QUESTIONS

F1. A pound equals a penny

Proof: $£1 = 100p = (10p)^2 = £(0.1)^2 = £0.01 = 1p$

F2. $2 = 3$

Proof: $2 \times 0 = 0$ so $2 = 0 \div 0$ Also, $3 \times 0 = 0$ so $3 = 0 \div 0$ Since $0 \div 0$, however it is defined, is clearly the same as $0 \div 0$, we must have $2 = 3$

F3. The perimeter of a square is four times its area

Proof: Choose units so that the side of the square is length 1. Then the perimeter equals 4 units and the area equals $1 \times 1 = 1$ unit. Thus, the perimeter of the square is four times its area.

F4. $0 = 1$

Proof: $0 = 0 + 0 + 0 + \cdots$ But $0 = 1 - 1$, so

$$0 = (1-1) + (1-1) + (1-1) + \cdots$$

So, by rearranging the brackets, we have

$$0 = 1 + (-1+1) + (-1+1) + (-1+1) + \cdots = 1 + 0 + 0 + 0 + \cdots = 1$$

F5. There is a living man with at least four heads

Proof: No man has three heads. Any living man has at least one head more than no man. Therefore, there is a living man with at least four heads.

MAIN QUESTIONS

M1. Any two real numbers are the same

Proof: Pick any three real numbers a, b and c. If $a^b = a^c$, then $b = c$ Therefore, since $1^x = 1^y$, we may deduce $x = y$ for any two real numbers x and y.

M2. $\infty = -1$

Proof: Let

$$x = 1 + 2 + 4 + 8 + \dots$$

Thus,

$$1 + 2x = 1 + 2(1 + 2 + 4 + \cdots) = 1 + 2 + 4 + 8 + \cdots = x$$

Thus, $1 + 2x = x$Rearranging this gives $x = -1$ However, x is also obviously infinite. Thus, $\infty = -1$

M3. All numbers are the same

Proof: Suppose that all numbers were not the same. Choose two numbers a and b which are not the same. Therefore one is bigger; we can suppose that $a > b$ Therefore, there is a positive number c such that $a = b + c$Therefore, multiplying sides by $(a - b)$gives

$$a(a - b) = (b + c)(a - b)$$

Expanding gives

$$a^2 - ab = ab - b^2 + ac - bc$$

Rearranging gives

$$a^2 - ab - ac = ab - b^2 - bc$$

Taking out a common factor gives

$$a(a - b - c) = b(a - b - c)$$

Dividing through gives $a = b$, therefore a and b could not have been different after all, hence all numbers are the same.

M4. All numbers are equal

Proof: Choose any two numbers a and b and let $a + b = s$

Thus, $(a + b)(a - b) = s(a - b)$

Thus, $a^2 - b^2 = sa - sb$

Thus, $a^2 - sa = b^2 - sb$

Thus, $a^2 - sa + s^2/4 = b^2 - sb + s^2/4$

Thus, $(a - s/2)^2 = (b - s/2)^2$

Thus, $a - s/2 = b - s/2$

Thus, $a = b$

E1. $3 = 0$

Proof: Consider the quadratic equation $x^2 + x + 1 = 0$Then, we can see that

$x^2 = -x - 1$Assuming that x is not zero (which it clearly isn't, from the equation) we can divide by x to give

$$x = -1 - \frac{1}{x}$$

Substitute this back into the x term in the middle of the original equation, so

$$x^2 + \left(-1 - \frac{1}{x}\right) + 1 = 0$$

This reduces to

$$x^2 = \frac{1}{x}$$

So, $x^3 = 1$, so $x = 1$is the solution. Substituting back into the equation for x gives

$$1^2 + 1 + 1 = 0$$

Therefore, $3 = 0$

E2. The smallest positive number is 1

Proof: Suppose that x is the smallest positive number. Clearly $x \leq 1$and also $x^2 > 0$ Since x is the smallest positive number, x^2 can't be smaller then x, so we must have $x^2 \geq x$. We can divide both sides of this by the positive number x to get $x \geq 1$ Since x is both less than or equal to 1 and greater than or equal to 1, x must equal 1. Thus the smallest positive number is 1.

E3. $1 = -1$

Proof: Clearly, $-1 = -1$and $\frac{1}{1} = \frac{-1}{-1}$Therefore, $-1 \times \frac{1}{1} = -1 \times \frac{-1}{-1}$Therefore, $\frac{-1\times1}{1} = \frac{-1\times-1}{-1}$Therefore, $\frac{-1}{1} = \frac{1}{-1}$Therefore, $\sqrt{\frac{-1}{1}} = \sqrt{\frac{1}{-1}}$ Therefore, $\frac{\sqrt{-1}}{1} = \frac{1}{\sqrt{-1}}$ Multipliying both sides by $\sqrt{-1} \times 1$gives $\sqrt{-1} \times \sqrt{-1} = 1 \times 1$Therefore, $-1 = 1$

E4. All cows in a field are the same colour
Proof : by induction on the number of cows.
Induction hypothesis: n cows in a field are the same colour, for all n=1,2,3,4,...
Initial Step: Clearly one cow in a field is the same colour as itself, so the induction hypothesis is true for n=1.
Inductive Step: Now suppose n is at least 1, we have n cows in a field F, and that the induction hypothesis has been proved for all fields containing at most n cows. By the induction hypothesis all the cows in F are the same colour. Now, remove any cow from F and put it to one side. Then take a cow from some other field and put it in field F. We again have n cows in field F, so by the induction hypothesis they are all the same colour. Finally, put back the first cow you thought of. We already know that it's the same colour as all the other cows in F, and so now we have n+1 cows in field F, and they're all the same colour. This completes the induction step!